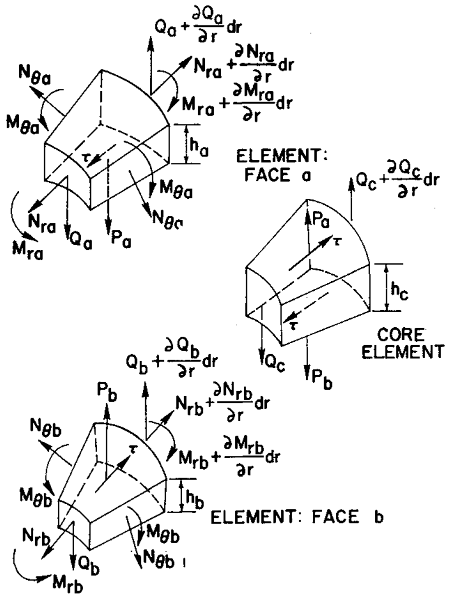

Fig. 2 Internal stress resultants of face and core elements

with T_a and T_b independent of r and z ($M_a^* = 0$, $M_b^* = 0$); thus,

$$f(r) = (C_1 r^2/2) \ln r + r^2/4(C_2 - 2C_1) + C_3 \ln r + C_4 + [2/(1 - v)(K_a + K_b)] \times (K_b T_a^* - K_a T_b^*)(r^2/4)$$

Let

$$\tfrac{1}{4}\{C_2 - 2C_1 + [2/(1 - v)(K_a + K_b)][(K_b T_a^* - K_a T_b^*)]\} = S$$

Therefore

$$f(r) = (C_1 r^2/2)(\ln r) + Sr^2 + C_3 \ln r + C_4$$

Boundary Conditions

For $r = 0$,

$$dw/dr = 0 \qquad u = 0$$

Therefore $C_1 = 0$,

$$1/r\, dw/dr - d^2w/dr^2 - r\, d^3w/dr^3 = 0 \qquad (Q_a + Q_b = 0)$$

For $r = R$,

$$w = 0$$

$$v dw/dr + R\, d^2w/dr^2 = 0 \qquad (M_r = 0,\ M^* = 0)$$

$$vu/R + du/dr = T^*/(1 - v)K \qquad (N_{ra} = 0,\ N_{rb} = 0;\ N_{ra} - N_{rb} = 0)$$

Therefore

$$(K_b K_a)(h_c/G_c)(D_a + D_b/h_0) d^2/dr^2(d^2w/dr^2 + 1/r\, dw/dr) = -(1/1 - v)(K_b T_a^* - K_a T_b^*)$$

Employing the method of variation of parameters and substituting the boundary conditions, it can be shown that the expression for the plate center deflection is

$$W_{r=0} = \{\alpha(T_a - T_b)(1 + v)(R^2)/[1 + \tfrac{1}{3}(h/h_0)^2](h_0)\} \times \{(\bar{I}_0 - 1)/\bar{X}^2 - [\tfrac{1}{2}(1 + v)][\bar{I}_0 - \bar{I}_1(1 - v)/\bar{X}]\} \times [1/(\bar{I}_0 - \bar{I}_1/\bar{X})]$$

where

$$\bar{X} = aR \qquad \bar{I}_0 = I_0(\bar{X}) \qquad \bar{I}_1 = I_1(\bar{X})$$

and

$$h_a = h_b = h \qquad K_a = K_b \qquad D_a = D_b \qquad T_a - T_b = \Delta T$$
$$a^2 = [1 + \tfrac{1}{3}(h/h_0)^2][6G_c(h_0/h)^2(1 - v^2)/E\, h\, (h_0 - h)]$$

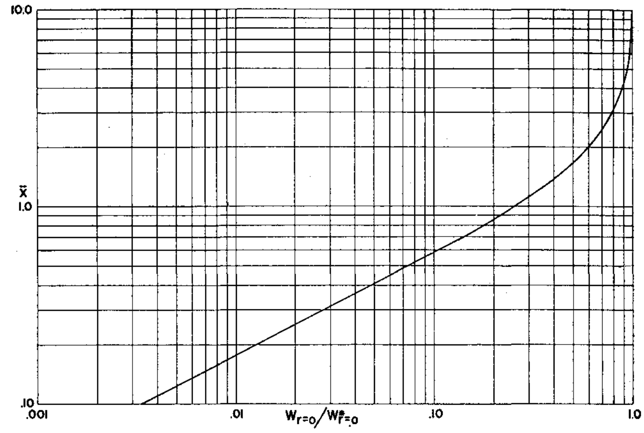

Fig. 3 Effect of core rigidity on deflection (nondimensionalized)

It can be shown that if $a \to \infty$ the expression for $W_{r=0}$ becomes

$$W_{r=0} \text{ with } a \to \infty = W_{r=0}^* = \alpha(\Delta T)\, R^2/2[1 + (\tfrac{1}{3})(h/h_0)^2](h_0)$$

Figure 3 is a plot of $W_{r=0}/W_{r=0}^*$ vs $\bar{X}$ for $v = 0.3$.

Bending Stresses: Particular Solution

$$M_r = -D(d^2w/dr^2 + v/r\, dw/dr) - 1/1 - v\, M^* \qquad (18^*)$$

$$M^* = 0 \text{ for particular case}$$

$$\sigma_r = 6M_r/h^2$$

It may be shown that

$$\sigma_r = \{E\alpha(\Delta T)/2(1 - v)(h_0/h)[1 + \tfrac{1}{3}(h/h_0)^2]\} \times \{[I_0 - \bar{I}_0 + (1 - v)(\bar{I}_1/\bar{X} - I_1/X)]/(\bar{I}_0 - \bar{I}_1/\bar{X})\}$$

Similarly,

$$M_\theta = -D(1/r\, dw/dr + v\, d^2w/dr^2) - 1/1 - v\, M^*$$

$$\sigma_\theta = 6M_\theta/h^2 \qquad (19^*)$$

$$\sigma_\theta = \{E\alpha(\Delta T)/2(1 - v)(h_0/h)[1 + \tfrac{1}{3}(h/h_0)^2]\} \times \{[vI_0 - \bar{I}_0 + (1 - v)(I_1/X + \bar{I}_1/\bar{X})]/(\bar{I}_0 - \bar{I}_1/\bar{X})\}$$

where

$$I_0 = I_0(X) \qquad \bar{I}_0 = I_0(\bar{X})$$
$$I_1 = I_1(X) \qquad \bar{I}_1 = I_1(\bar{X})$$
$$x = ar \qquad \bar{X} = aR$$

A Closed Form Solution of the Relativistic Differential Equation for Planetary Motion

HENRY L. CROWSON*
International Business Machines Corporation, Bethesda, Md.

1. Introduction

AN outline of the derivation of Einstein's general relativistic differential equations for the motion of a particle in a gravitational field has been given by Lindsay and Margenau[3] and Sokolnikoff.[6] A detailed exposition of the derivation of these equations has been given by the author.[1]

Received December 20, 1962.
* Staff Mathematician, Federal Systems Division.

In this paper it will be shown that a solution of certain of the relativistic differential equations for planetary motion can be expressed in closed form in terms of elliptic functions. A comparison of the solution so obtained with a solution of the classical or Newtonian equation for planetary motion will be given.

2. Sketch of Derivation of Equations

Consider a particle moving in a gravitational field along an arc of length s, where $s = s(x,y,z,t)$. A fundamental principle in the derivation is that s is stationary, so that the variation of s vanishes. Symbolically,

$$\delta \int ds = 0 \tag{1}$$

Apply Hamilton's principle[3] to Eq. (1) to obtain

$$\delta \int (c^2 - L)dt = 0 \tag{2}$$

where L is the ordinary Langrangian function, and c is the velocity of light in free space.

Considerable algebraic manipulation applied to Eq. (2) yields the following set of four differential equations of the second order:

$$\frac{d^2x_i}{ds^2} + \sum_{j,k=1}^{4} \left\{ \begin{matrix} j\ k \\ i \end{matrix} \right\} \frac{dx_j}{ds} \cdot \frac{dx_k}{ds} = 0 \ (i = 1, 2, 3, 4) \tag{3}$$

where the three-index symbol

$$\left\{ \begin{matrix} j\ \ k \\ i \end{matrix} \right\}$$

is used in the fourth-order Riemann-Christoffel mixed covariant-contravariant tensor to denote the following expression:

$$\left\{ \begin{matrix} j\ \ k \\ i \end{matrix} \right\} = \sum_{r=1}^{4} \frac{1}{2} g^{(ri)} \left[\frac{\partial g_{jr}}{\partial x_k} + \frac{\partial g_{sr}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_r} \right] \tag{4}$$

and $g^{(ri)}$ is defined in Refs. 3 and 6. In Eq. (3), one should note that the space-time coordinates x, y, z, and t have been mapped into the generalized coordinates x_1, x_2, x_3, and x_4, respectively.

Further reduction of Eq. (3) yields the following:

$$(d^2w/d\phi^2) + w = (3mw^2) + m/C_1^2 \tag{5}$$

and

$$d\phi/ds = C_1 w^2 \tag{6}$$

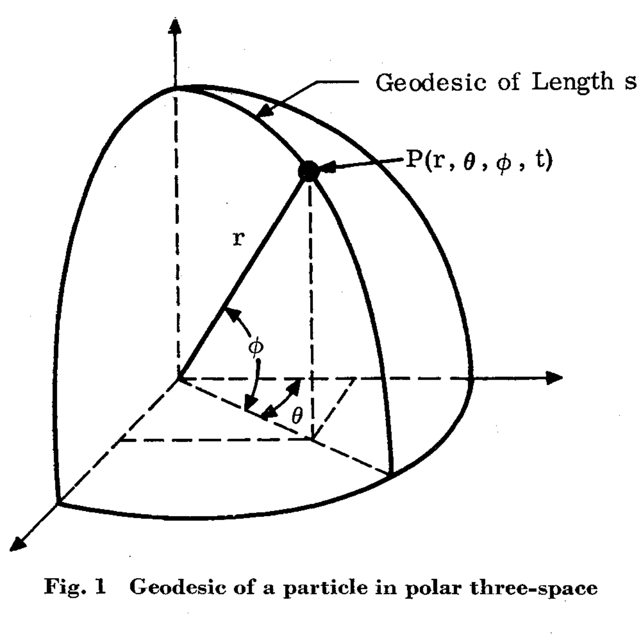

Fig. 1 Geodesic of a particle in polar three-space

where

w = $1/r$
r = polar radius of the gravitational field
m = GM
G = gravitational constant
M = mass of the body producing the gravitational field
C_1 = an arbitrary constant of integration
s = distance along a geodesic in a four dimensional manifold
ϕ = angle in polar three-space

One should note in Eqs. (5) and (6) that the geodesic of path length s now is represented in polar three-space and time t. That is, $s = s(r,\theta,\phi,t)$. A spatial representation of s is given in Fig. 1.

Equations (5) and (6) will determine the geodesic of a particle in a gravitational field, and Eq. (5) now will be solved.

3. Solution of the Relativistic Differential Equation

Equation (5) immediately can be put in the form

$$\frac{(d^2w/d\phi^2) - (m/C_1^2)}{3m} = w^2 - (w/3m) \tag{7}$$

Completing the square in w allows Eq. (7) to be written

$$d^2w/d\phi^2 = (1/12m)[(6mw - 1)^2 - 1] + (m/C_1^2) \tag{8}$$

Now make the linear transformation:

$$3z = 6mw - 1 \tag{9}$$

Then in terms of z, Eq. (8) is written

$$d^2z/d\phi^2 = (9z^2/6) + [(2m^2/C_1^2) - \tfrac{1}{6}] \tag{10}$$

Multiply both members of Eq. (10) by $dz/d\phi$ and integrate in accordance with[2]

$$\int \frac{d^2z}{d\phi^2}\left(\frac{dz}{d\phi} \cdot d\phi\right) = \frac{9}{6}\int z^2dz + \left(\frac{2m^2}{C_1^2} - \frac{1}{6}\right)\int dz + C_2$$

Thus,

$$(dz/d\phi)^2 = z^3 + [(4m^2/C_1^2) - \tfrac{1}{3}]z + C_2 \tag{11}$$

where C_2 is another arbitrary constant of integration.

For convenience, write Eq. (11) as follows:

$$(dz/d\phi)^2 = z^3 + pz + C_2 \tag{12}$$

where

$$p = (4m^2/C_1^2) - \tfrac{1}{3} \tag{13}$$

Cardan's method of factoring a cubic expression[7] permits the right member of Eq. (12) to be factored as follows:

$$(dz/d\phi)^2 = (z - a_1)(z - a_2)(z - a_3) \tag{14}$$

where, in trigonometric form,

$$a_1 = 2(-p/3)^{1/2}\cos(\alpha/3) \tag{15}$$

$$a_2 = 2(-p/3)^{1/2}\cos[(\alpha + 2\pi)/3] \tag{16}$$

$$a_3 = 2(-p/3)^{1/2}\cos[(\alpha + 4\pi)/3] \tag{17}$$

In Eqs. (15–17),

$$\alpha = \cos^{-1}[3C_2(3)^{1/2}/2p(-p)^{1/2}] \tag{18}$$

hence p must be less than zero. From Eq. (13), $p < 0$ implies $C_1^2 > 12\ m^2$, and since C_1 is an arbitrary constant of integration, it is easy to make $p < 0$. Further, α is taken in the first or second quadrant according as C_2 is negative or positive. When $C_2 = 0$, choose $\alpha = \pi/2$. Also, $p < 0$ and $\alpha \neq 2k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$ guarantees distinctness of a_1, a_2, and a_3, so that, without loss of generality, write $a_1 < a_2 < a_3$.

Taking the positive square root of both members of Eq. (14), separating variables, and integrating gives

$$\phi = \int \frac{dz}{[(z - a_1)(z - a_2)(z - a_3)]^{1/2}} + C_3 \quad (19)$$

From the transformation given by Eq. (9) and from the definition of w, it follows that

$$z = (2m/r) - \tfrac{1}{3}$$

and hence $z \rightarrow \infty$ as $r \rightarrow 0$. The following integral equation over the entire gravitational field thus emerges:

$$\phi(z) = \int_z^\infty \frac{dz}{[(z - a_1)(z - a_2)(z - a_3)]^{1/2}} \quad (20)$$

$$a_1 < a_2 < a_3 < z$$

In Eq. (20), make the following transformations in accordance with[4]

$$\sin^2\gamma = (a_3 - a_1)/(z - a_1) \quad (21)$$

and

$$k^2 = (a_2 - a_1)/(a_3 - a_1) \quad (22)$$

From Eq. (21),

$$(z - a_1) = (a_3 - a_1) \csc^2\gamma \quad (23)$$

and

$$-2(z - a_1) \cot\gamma d\gamma = dz \quad (24)$$

To evaluate $z - a_3$ in terms of γ, consider Fig. 2. From Fig. 2

$$z - a_3 = (z - a_1) \cos^2\gamma \quad (25)$$

Combining Eqs. (21) and (22) gives

$$1 - k^2 \sin^2\gamma = (z - a_2)/(z - a_1)$$

Hence

$$z - a_2 = (z - a_1)(1 - k^2 \sin^2\gamma) \quad (26)$$

Substitute Eqs. (23–26) into Eq. (20) to get

$$\phi(\gamma) = \int_\gamma^0 \frac{-2(z - a_1) \cot\gamma \, d\gamma}{\{[(a_3 - a_1) \csc^2\gamma][(z - a_1)(1 - k^2 \sin^2\gamma)][(z - a_1) \cos^2\gamma]\}^{1/2}}$$

or

$$\phi(\gamma) = \frac{2}{(a_3 - a_1)^{1/2}} \int_0^\gamma \frac{d\gamma}{(1 - k^2 \sin^2\gamma)^{1/2}} \quad (27)$$

$$0 < k < 1$$
$$0 < \gamma < \pi/2$$

where γ corresponds to z by virtue of Eq. (21).

The right member of Eq. (27) is an elliptic integral of the first kind,[5] so that it may be written

$$\phi(\gamma) = [2/(a_3 - a_1)^{1/2}]F(k, \gamma) \quad (28)$$

From Eqs. (9, 21, and 22), Eq. (28) is written in terms of r thus:

$$\phi(r) = \frac{2}{(a_3 - a_1)^{1/2}} F\left\{\left[\frac{a_2 - a_1}{a_3 - a_1}\right]^{1/2}, \sin^{-1}\left[\frac{3r\,(a_3 - a_1)}{6m - r(3a_1 + 1)}\right]^{1/2}\right\} \quad (29)$$

Equation (29) is the required solution of Eq. (5).

4. Solution of the Newtonian Differential Equation

For comparison purposes, display the Newtonian equations that determine the geodesic of a particle in a gravitational field

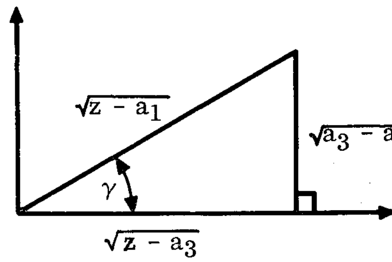

Fig. 2 Evaluating $z - a_3$ in terms of γ

$$(d^2w/d\phi^2) + w = m/C_1^2 \quad (30)$$

and

$$d\phi/ds = C_1 w^2 \quad (31)$$

To solve Eq. (30), let $u = w - m/C_1^2$; then write

$$(d^2u/d\phi^2) + u = 0 \quad (32)$$

A general solution of Eq. (32) is

$$u = C_3 \cos\phi + C_4 \sin\phi \quad (33)$$

In Eq. (33), let $C_3 = C_5 \cos \phi_1$ and $C_4 = C_5 \sin\phi_1$; then substitute into Eq. (33) to get

$$u = C_5(\cos\phi \cos\phi_1 + \sin\phi \sin\phi_1) \quad (34)$$

In view of the expressions for u and w, Eq. (34) can be written

$$r = C_1^2/m[1 + \epsilon \cos(\phi - \phi_1)] \quad (35)$$

where ϵ is the eccentricity of the path of a particle in a gravitational field and is defined by the equation $\epsilon = C_1^2C_5/m$. If $0 \leq \epsilon < 1$, the path is an ellipse.

Equation (35) may be written in the following alternate form:

$$\phi(r) = \cos^{-1}[(C_1^2 - mr)/\epsilon mr] + \phi_1 \quad (36)$$

A graph of Eq. (35) when $0 < \epsilon < 1$ is given in Fig. 3.

References

[1] Crowson, H. L., "Derivation of the relativistic differential equations for planetary motion," IBM Space Systems Center unpublished manuscript, pp. 11–22 (1960).

[2] Greenhill, A. G., *The Application of Elliptic Functions* (Dover Publications Inc., New York, 1957), p. 26.

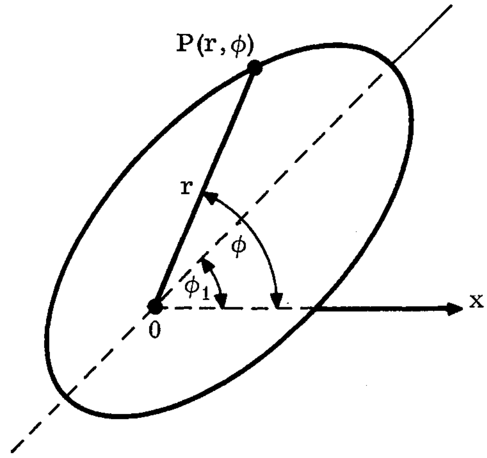

Fig. 3 Geodesic of a particle in Newtonian coordinates

[3] Lindsay, R. B. and Margenau, H., *Foundations of Physics* (Dover Publications Inc., New York, 1957), pp. 361–374.

[4] MacRobert, T. M., *Functions of a Complex Variable* (Macmillian Co., New York, 1954), 4th ed., p. 330.

[5] Reddick, H. W. and Miller, H. W., *Advanced Mathematics for Engineers* (John Wiley and Sons Inc., New York, 1951), 3rd ed., p. 131.

[6] Sokolnikoff, I. S., *Tensor Analysis* (John Wiley and Sons Inc., New York, 1951), pp. 277–283.

[7] Uspensky, J. V., *Theory of Equations* (McGraw-Hill Book Co. Inc., New York, 1948), p. 92.

Stresses in an Infinite Elastic Slab of Nonhomogeneous Transversely Isotropic Material

TARA GHOSH*
Basanti Debi College, Calcutta, India

THE object of this paper is to find the displacement and stresses in an infinite slab of transversely isotropic nonhomogeneous material that has a symmetrical distribution of shearing stresses over a circular area on the face $z = 0$, whereas the face $z = b$ is rigidly fixed. The elastic moduli are supposed to vary exponentially with the depth.

The origin is taken on the free surface $z = 0$ of the infinite slab, and the axis of z is drawn into the body at a right angle to this plane. Assuming the axis of z to be the axis of symmetry in a transversely isotropic material, the following stress-strain relations are obtained:[1]

$$\begin{aligned} p_{xx} &= e_{11}e_{xx} + e_{12}e_{yy} + e_{13}e_{zz} \\ p_{yy} &= e_{12}e_{xx} + e_{11}e_{yy} + e_{13}e_{zz} \\ p_{zz} &= e_{13}(e_{xx} + e_{yy}) + e_{33}e_{zz} \\ p_{yz} &= e_{44}e_{yz} \\ p_{zx} &= e_{44}e_{zx} \\ p_{xy} &= [(e_{11} - e_{12})/2]e_{xy} \end{aligned} \tag{1}$$

The equations of equilibrium in absence of body forces are

$$\begin{aligned} \frac{\partial}{\partial x} p_{xx} + \frac{\partial}{\partial y} p_{xy} + \frac{\partial}{\partial z} p_{xz} &= 0 \\ \frac{\partial}{\partial x} p_{xy} + \frac{\partial}{\partial y} p_{yy} + \frac{\partial}{\partial z} p_{yz} &= 0 \\ \frac{\partial}{\partial x} p_{xz} + \frac{\partial}{\partial y} p_{yz} + \frac{\partial}{\partial z} p_{zz} &= 0 \end{aligned} \tag{2}$$

The components of strain are

$$\begin{gathered} e_{xx} = \partial u/\partial x \qquad e_{yy} = \partial v/\partial y \qquad e_{zz} = \partial w/\partial z \\ e_{xy} = (\partial u/\partial y) + (\partial v/\partial x) \qquad e_{yz} = (\partial w/\partial y) + (\partial v/\partial z) \\ e_{zx} = (\partial u/\partial z) + (\partial w/\partial x) \end{gathered} \tag{3}$$

To solve the problem, assume

$$u = -(\partial\phi/\partial y) \qquad v = \partial\phi/\partial x \qquad w = 0 \tag{4}$$

where ϕ is a function of coordinates.

Then the stress components are obtained as

$$\begin{aligned} p_{xx} &= -A(\partial^2\phi/\partial x\partial y) & p_{yz} &= G(\partial^2\phi/\partial x\partial z) \\ p_{yy} &= A(\partial^2\phi/\partial x\partial y) & p_{zx} &= -G(\partial^2\phi/\partial y\partial z) \\ p_{zz} &= 0 & p_{xy} &= (A/2)[(\partial^2\phi/\partial x^2) - (\partial^2\phi/\partial y^2)] \end{aligned} \tag{5}$$

in which $A = e_{11} - e_{12}$ and $G = e_{44}$. Assuming $A = A_0e^{-mz}$ and $G = G_0e^{-mz}$ and using these relations in (5) and (2), one finds that the third equation is satisfied identically and that the other two give

$$\begin{aligned} \frac{\partial}{\partial y}\left[\frac{A_0}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + G_0\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right)\right] &= 0 \\ \frac{\partial}{\partial x}\left[\frac{A_0}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + G_0\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right)\right] &= 0 \end{aligned} \tag{6}$$

These are satisfied if

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k_0^2\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right) = 0 \tag{7}$$

where $k_0^2 = 2G_0/A_0$.

Transforming this equation into cylindrical coordinates (r,θ,ϕ), one obtains for an axially symmetric distribution of stresses

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + k_0^2\left(\frac{\partial^2\phi}{\partial z^2} - m\frac{\partial\phi}{\partial z}\right) = 0 \tag{8}$$

To solve this equation, assume that $\phi = R(r)Z(z)$; then (8) reduces to

$$(d^2R/dr^2) + (1/r)(dR/dr) + \alpha^2R = 0 \tag{9}$$

and

$$k_0^2(d^2Z/dz^2) - mk_0^2(dZ/dz) - \alpha^2Z = 0 \tag{10}$$

Therefore, the solution of (8) is

$$\phi = \int_0^\infty [A_1e^{n_1z} + B_1e^{n_2z}]J_0(\alpha r)d\alpha \tag{11}$$

where $n_{1,2} = [mk_0 \pm (m^2k_0^2 + 4\alpha^2)^{1/2}]/2k_0$.

The components of displacement and stress are obtained as

$$\begin{aligned} u_\theta &= -\int_0^\infty \alpha[A_1e^{n_1z} + B_1e^{n_2z}]J_1(dr)d\alpha \\ p_{\theta z} &= -G_0\int_0^\infty \alpha[A_1n_1e^{(n_1-m)z} + B_1n_2e^{(n_2-m)z}]J_1(\alpha r)d\alpha \\ p_{r\theta} &= \frac{A_0}{2}\left\{\int_0^\infty \alpha^2\,[A_1e^{(n_2-m)z} + B_1e^{(n_2-m)z}]\right\}J_2(\alpha r)d\alpha \end{aligned} \tag{12}$$

To obtain the boundary conditions, assume the shearing stress at the boundary $z = 0$ to be acting within a circular area of a radius a and to be given by the functions

$$\begin{aligned} u_\theta &= 0 \text{ on } z = b \\ p_{\theta z} &= 0 \text{ on } z = 0 \qquad a < r \\ p_{\theta z} &= F(r) = Qr \text{ on } z = 0 \qquad 0 \leqslant r \leqslant a \end{aligned} \tag{13}$$

where Q is a constant.

Table 1

	1	2	3	4	5
r	$0.1a$	$0.2a$	$0.3a$	$0.4a$	$0.5a$
$(u_\theta)_{z=0}G_0/Q$	−0.0008	−0.0015	−0.0024	−0.0031	−0.0038

Received by IAS August 22, 1962. The author is grateful to B. Sen for his kind help and valuable guidance.

* Lecturer in Mathematics.

[1] Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, New York, 1927), 4th ed., p. 160.